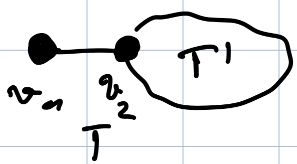
Ex 1 (1) Let n be the number of vertices of T.

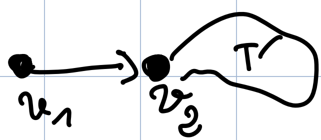
If $n=1$, $T = \bullet$ only one orientation

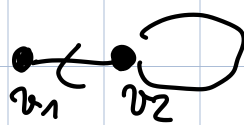
(2) If Q_1, Q_2 are two orientations of T. We choose v a leaf of T



If v_1, v_2 has the same orientation in Q_1, Q_2 we erase it to get Q_1', Q_2' and apply induction

Otherwise



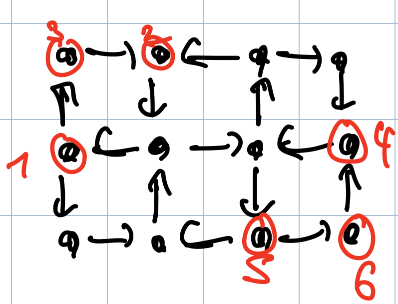


1st do a mutation at v_1, then apply induction.

(2)
$$
\begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r, "\mu'_2"'] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r] \arrow[dr] & 2 \arrow[d] \\
 & 3
\end{tikzcd}
$$
Not a tree

$$
\begin{tikzcd}
1 \arrow[dr] & 2 \arrow[l] \\
 & 3 \arrow[u]
\end{tikzcd}
$$

(3)



Ex 2 (1) $(1, x_1) \overset{\mu_1}{\underset{\longleftarrow}{\longrightarrow}} (1, \frac{2}{x_1})$ $\quad x_1' x_1 = 2 \Rightarrow x_1' = \frac{2}{x_1}$

$$x_1' x_1'' = 2 \Rightarrow x_1'' = 2 \times \frac{x_1}{2} = x_1$$

$$\begin{tikzcd}
\overset{x_1}{1} \arrow[r] & \overset{x_2}{2} \arrow[rr, "\mu_2"] \arrow[d, "\mu_1"'] & & \overset{x_1}{1} & \overset{\frac{1+x_1}{x_2}}{2} \arrow[l] \arrow[d, "\mu_1"] \\
\underset{\frac{1+x_2}{x_1}}{1} & \underset{x_2}{2} \arrow[l] \arrow[r, "\mu_2"] & \underset{\frac{1+x_2}{x_1}}{1} \arrow[r] & \underset{\frac{1+x_1+x_2}{x_1x_2}}{2} \arrow[r, "\mu_1"] & \overset{\frac{1+x_1+x_2}{x_1x_2}}{1} \to \overset{\frac{1+x_1}{x_2}}{2}
\end{tikzcd}$$

$$x_2' x_2 = 1 + x_1$$

$$x_1 x_1' = \frac{1+x_1}{x_2} + 1$$

$$x_1' = \frac{1+x_1+x_2}{x_1x_2}$$

$$x_2 x_2' = 1 + \frac{1+x_2}{x_1}$$
$$= \frac{1+x_1+x_2}{x_1}$$

$$\begin{tikzcd}
\overset{x_1}{1} \arrow[r] & \overset{x_2}{\boxed{2}} \arrow[r, "\mu_1"'] & \overset{\frac{1+x_2}{x_1}}{1} & \overset{x_2}{\boxed{2}} \arrow[l]
\end{tikzcd}$$

$$x_1 x_1' = 1 + x_2$$
$$x_1' = \frac{1+x_2}{x_1}$$
$$x_1'' = \frac{(x_2+1)\times x_1}{x_2+1}$$

- $$\begin{tikzcd}
\underset{x_2}{\boxed{2}} \arrow[r] & \underset{x_1}{1} \arrow[r] \arrow[d, "\mu_1"] & \underset{x_3}{\boxed{3}} \\
\boxed{2} & 1 \arrow[l] & \boxed{3} \arrow[l]
\end{tikzcd}$$

$$x_1 x_1' = x_2 + x_3 \Rightarrow x_1' = \frac{x_2+x_3}{x_1}$$

$x_2 \quad \frac{x_2+x_3}{x_1} \quad x_3 \qquad\qquad x_1$

$\big\downarrow \mu'_1$

$\boxed{2} \to 1 \to \boxed{3}$ (1: x_1) $\qquad x'_1 x''_1 = x_2 + x_3 \Rightarrow x''_1 = x_1$

$\boxed{2} \leftarrow 1 \to \boxed{3}$ (x_2, x_1, x_3) $\qquad x'_1 = \frac{1+x_2x_3}{x_1}$

$\big\downarrow$

$\boxed{} \to 1 \leftarrow \boxed{}$ (1: $\frac{1+x_2x_3}{x_1}$) $\qquad x''_1 = \frac{1+x_2x_3 \times x_1}{1+x_2x_3}$

- $1 \to 2 \to \boxed{3}$ (x_1, x_2, x_3)

$\mu_1 \swarrow \qquad \searrow \mu_2$

$1 \leftarrow 2 \to \boxed{3}$ ($\frac{1+x_2}{x_1}$, x_2, x_3)

$\big\downarrow \mu_2$

$1 \to 2 \leftarrow \boxed{3}$ ($\frac{1+x_2}{x_1}$, $\frac{x_1+x_3+x_2x_3}{x_1x_2}$, x_3)

$\big\downarrow \mu_1$

$1 \leftarrow 2 \leftarrow \boxed{3}$ ($\frac{x_1+x_3}{x_2}$, $\frac{x_1+x_3+x_2x_3}{x_1x_2}$, x_3)

$1 \leftarrow 2 \leftarrow 3$ (x_1, $\frac{x_1+x_3}{x_2}$, x_3)

$\big\downarrow \mu_1$

$1 \to 2 \quad 3$ ($\frac{x_1+x_3+x_2x_3}{x_2x_1}$, $\frac{x_1+x_3}{x_2}$, x_3)

$=$ Modulo Umbenennung

$x_1x'_1 = \frac{x_1+x_3}{x_2} + x_3 = \frac{x_1+x_3+x_2x_3}{x_2}$

(2) $\begin{array}{cc} \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$ $\quad \underset{x_1}{1} \overset{2}{-\!-} \underset{x_2}{2}$ type B_2

<u>recall matrix mutation:</u>

$$\mu_k : \begin{cases} b'_{ij} = -b_{ij} & \text{if } i=k \text{ or } j=k \\ b_{ij} + b_{ik}b_{kj} & b_{ik}>0,\ b_{kj}>0 \\ b_{ij} - b_{ik}b_{kj} & \text{both negative} \\ b_{ij} & \text{o/w} \end{cases}$$

$$x_k x_k' = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}$$

b_{12} b_{22}

$$\begin{tikzcd}[column sep=small]
 & \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} (x_1, x_2) \arrow[dl, "\mu_1"'] \arrow[dr, "\mu_2"] & \\
\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, \left(\frac{1+x_2^2}{x_1}, x_2\right) \arrow[d, "\mu_2"] & & \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \left(x_1, \frac{x_1+1}{x_2}\right) \arrow[d, "\mu_1"] \\
\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \left(\frac{1+x_2^2}{x_1}, \frac{x_1+x_2^2+1}{x_1x_2}\right) \arrow[dr, "\mu_1"'] & & \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \left(\frac{1+2x_1+x_1^2+x_2^2}{x_1x_2^2}, \frac{x_1+1}{x_2}\right) \arrow[dl, "\mu_2"] \\
 & \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \left(\frac{1+2x_1+x_1^2+x_2^2}{x_1x_2^2}, \frac{x_1+x_2^2+1}{x_1x_2}\right) &
\end{tikzcd}$$

(3) $1 \rightrightarrows 2$

$\big\downarrow \mu_1$ u μ_2

$1 \leftleftarrows 2$

$$\begin{array}{ccccccc} u_0 & u_1 & & u_2 = \frac{1+x_2^2}{x_1} & x_2 & & u_2 = \left(\frac{1+x_2^2}{x_1}\right) \\ x_1 & x_2 & & & & & \\ 1 & \rightrightarrows 2 & \xrightarrow{\mu_1} & 1 & \leftleftarrows 2 & \xrightarrow{\mu_2} & 1 \rightrightarrows 2 \end{array}$$

$x_2' x_2 = u_2^2 + 1$

$u_3 = \frac{u_2^2+1}{u_1}$

$$u_{n+1} = \frac{u_n^2+1}{u_{n-1}}$$

$u_0 = 1 = u_1 \quad u_2 = 2 \quad u_3 = 5 \quad u_4 = 13$
$u_5 = 34 \ldots$

We have $0 < u_1 < u_2 < u_3$, by induction we prove $u_n < u_{n+1}$

$$u_{n+1} = \frac{u_n^2+1}{u_{n-1}} > \frac{u_{n-1}u_{n+1}}{u_{n-1}} > u_n \qquad + \underbrace{\frac{1}{u_{n-1}}}_{>0} > u_n$$

so $(u_n)_n$ is strictly increasing so it has ∞-many distinct values.

Fibo $\underline{1}, 1, \underset{u_{n-1}}{\underline{2}}, 3, \underset{u_n}{\underline{5}}, 8, \underline{13}, 21, \underline{34}, 55, \underline{89}$

⊙ $f_n = f_{n-1} + f_{n-2}$ gives Fibonacci numbers

(b) $u_n f_n = f_{n-1} + u_{n-1}$ $\qquad\qquad$ $u_{n-1} = f_{n-2} + u_{n-2}$

$$= 2u_{n-1} + f_{n-2}$$

$$= \boxed{3u_{n-1} - u_{n-2}}$$

By induction we prove that u_n satisfies this reduction formula

$u_n = 3u_{n-1} - u_{n-2}$ by induction so

$$u_n^2 = 3u_{n-1}u_n - u_{n-2}u_n \qquad\qquad u_{n+1} = \frac{u_n^2+1}{u_{n-1}}$$

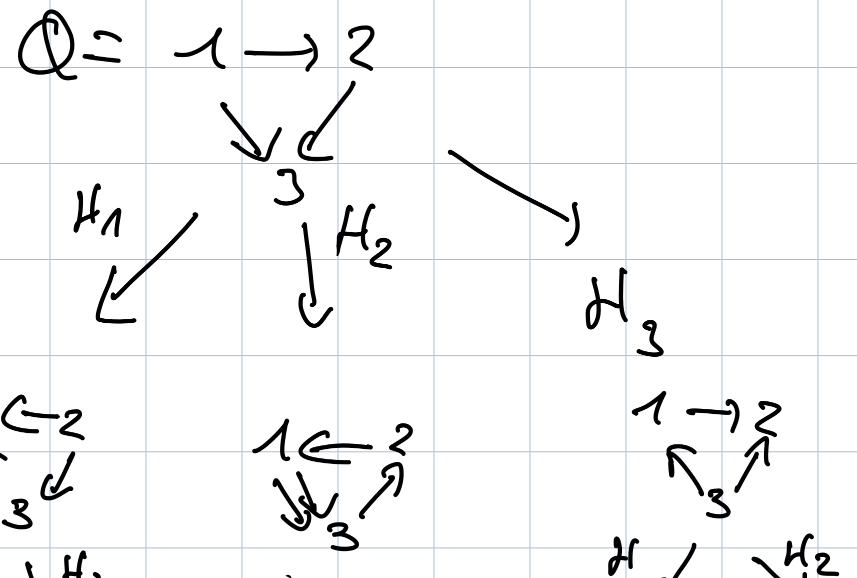
$$= 3u_{n-1}u_n - u_{n-1}^2 - 1$$

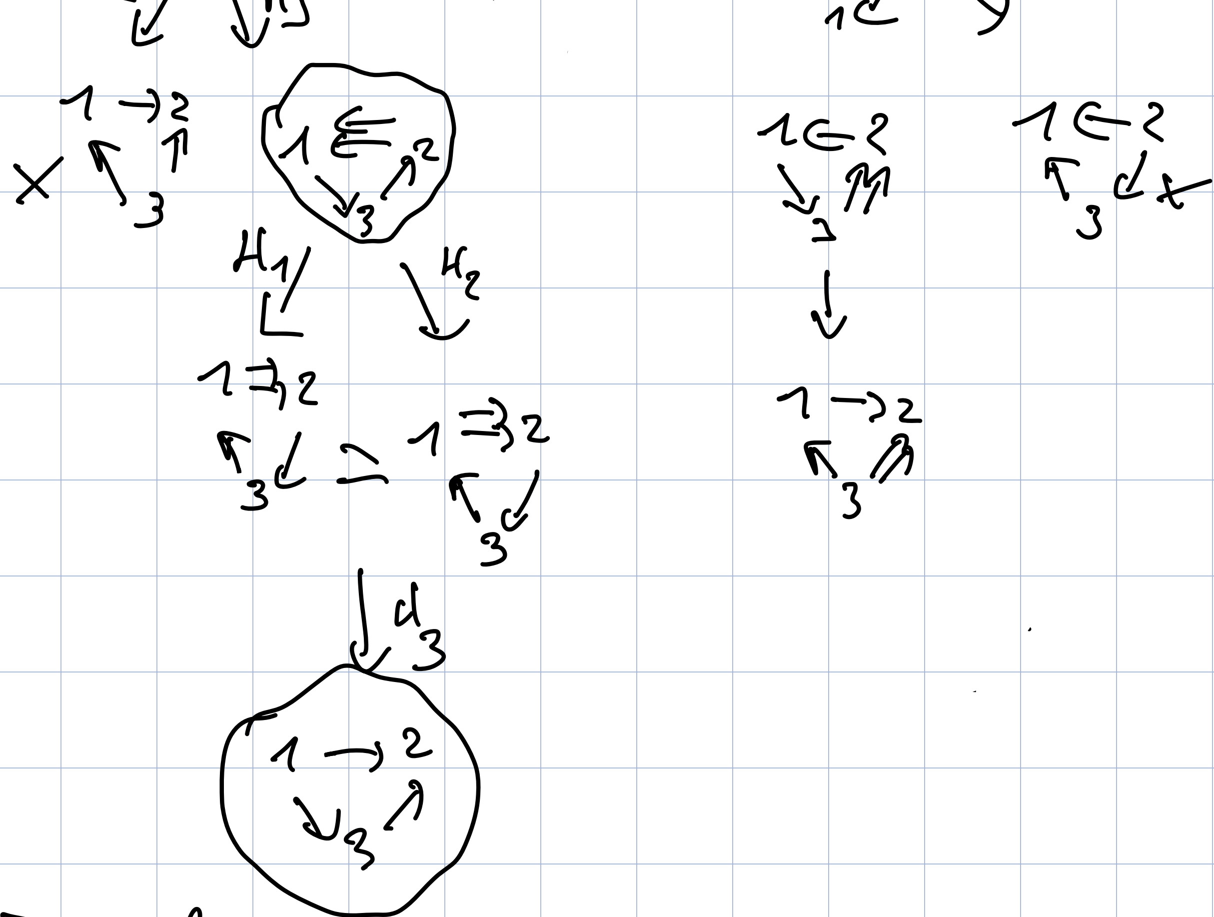
$$\Rightarrow u_n^2 + 1 = u_{n-1}(3u_n - u_{n-1})$$

$$\Rightarrow \frac{u_n^2+1}{u_{n-1}} = 3u_n - u_{n-1}$$

so $u_{n+1} = 3u_n - u_{n-1}$ -

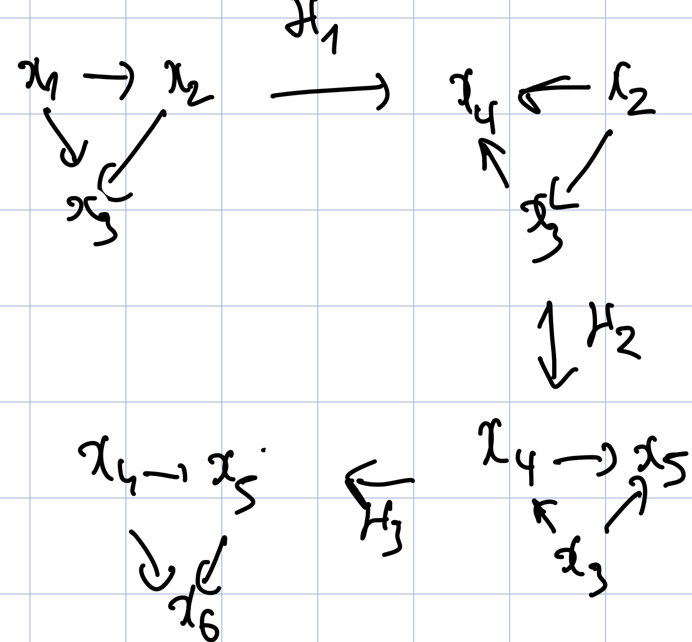
(4) (a)





Find all 12.

(b)



Formula: $x_1 x_4 = 1 + x_2 x_3$ by symmetry we get

$$\boxed{x_n x_{n+3} = 1 + x_{n+1} x_{n+2}}$$

As before not even clear that it is a sequence of integers

$x_1 = 1 = x_2 = x_3 \quad x_4 = 2, \quad x_5 = 3, \quad x_6 = 7, \quad x_7 = 11,$

as before we see that the sequence seems strictly increasing

$$x_{n+3} = \frac{1 + x_{n+1} x_{n+2}}{x_n} > \frac{1 + x_n x_{n+2}}{x_n} = \frac{1}{x_n} + x_{n+2} > x_{n+2}$$

so we are done.

we also see an inductive pattern:

$$1, 1, 1, 2, 3, 7, 11, 26, 41, 97, \dots$$

$= 4 \cdot 3 - 1$

$a_n = 4a_{n-2} - a_{n-4}$.

$\mu_1 k$

Ex 3 (1)
$$\begin{array}{c|c} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 4 \\ -1 & 0 \\ 1 & -3 \end{pmatrix} \end{array}$$

$1 \xrightarrow{4} 2$, $2 \to \boxed{3}$, $\boxed{3} \to 1$

$x_1 x_1' = x_2 + x_3$

$$\begin{pmatrix} 0 & \boxed{4} \\ -1 & 0 \\ \boxed{1} & -3 \end{pmatrix} (x_1, x_2, x_3) \xrightarrow{\mu_1} \begin{pmatrix} 0 & -4 \\ +1 & 0 \\ -1 & 1 \end{pmatrix} \left(\frac{x_2 + x_3}{x_1}, x_2, x_3\right)$$

$1 \xrightarrow{4} 2$, $2 \to \boxed{3}$, $\boxed{3} \to 1$ $\qquad\qquad$ $1 \xleftarrow{4} 2$, $1 \to \boxed{3}$, $\boxed{3} \to 2$

$\downarrow \mu_2$

$$\begin{pmatrix} 0 & 4 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \left(\frac{x_2+x_3}{x_1}, \frac{\left(\frac{x_2+x_3}{x_1}\right)^4 + x_3}{x_2}, x_3 \right)$$

So starting from $(3,-1,10)$ we get $(5,-1,16)$
$(5,641,16)$

By the Laurent phenomenon we have that the cluster variables are Laurent polynomials in any extended cluster (= initial variables).
Moreover the frozen variables do not appear in the denominators.

<u>Here</u> $(3,-1,16) \to$ each specialization is an integer or an integer divided by a power of 3.

Using $(5,-1,16) \to$ " " power of 5

$$\frac{n}{3^a} = \frac{n'}{5^b} \Rightarrow 3^a n' = 5^b n \quad \text{Gauss} \Rightarrow 3^a / n \text{ so } \underline{\text{integer}}$$

We only get integers as specialization of each cluster variable.

Let us continue the mutations:

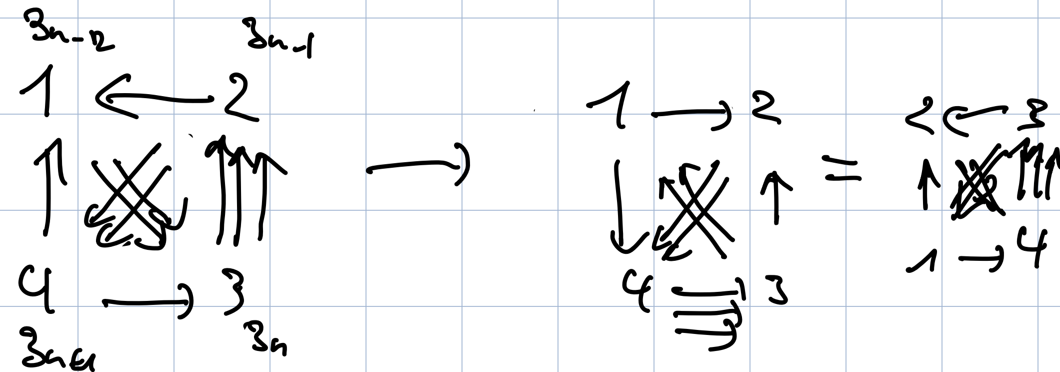
$$\begin{pmatrix} 0 & 4 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} (5, -641, 16) \xrightarrow{H_1} \begin{pmatrix} 0 & -4 \\ 1 & 0 \\ 0 & -1 \end{pmatrix} (-128, -641, 16)$$

$$\xrightarrow[H_2]{} \begin{pmatrix} 0 & 4 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \left(-128, \frac{2^{32}+1}{641}, 16\right)$$

$$x_2 x_2' = x_1^4 x_3 + 1 = (2^7)^4 2^4 + 1$$

$$= 2^{32} + 1$$

Donc $x_2' = \dfrac{2^{32}+1}{641} \in \mathbb{Z}$.

(2) $Q =$ 

So H_1 amounts to apply a rotation by an angle $-\frac{\pi}{2}$

Then $H_4 H_3 H_2 H_1$ gives back Q

So starting with (u_1, u_2, u_3, u_4) we can construct a sequence by applying this successive mutations.

We have $\boxed{z_{n+2}z_{n-2} = z_n^2 + z_{n-1}z_{n+1}}$

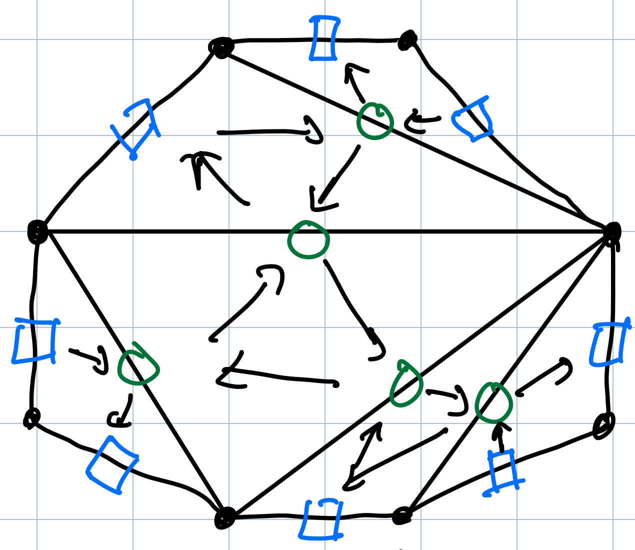
(c) Starting with the conditions $(1,1,1,1)$, the Laurent phenomenon implies that (z_n) is a sequence of integers.

Starts with $1,1,1,1,2,3,7,23,59,\dots$

It was conjectured by Somos in 89 that this indeed a sequence of integers.
Proved in 90 Janice Malouf not an obvious proof!, there is also a bijective proof (~2000 Propp)

Ex 4 Triangulations

P_m convex m-gon with a triangulation T



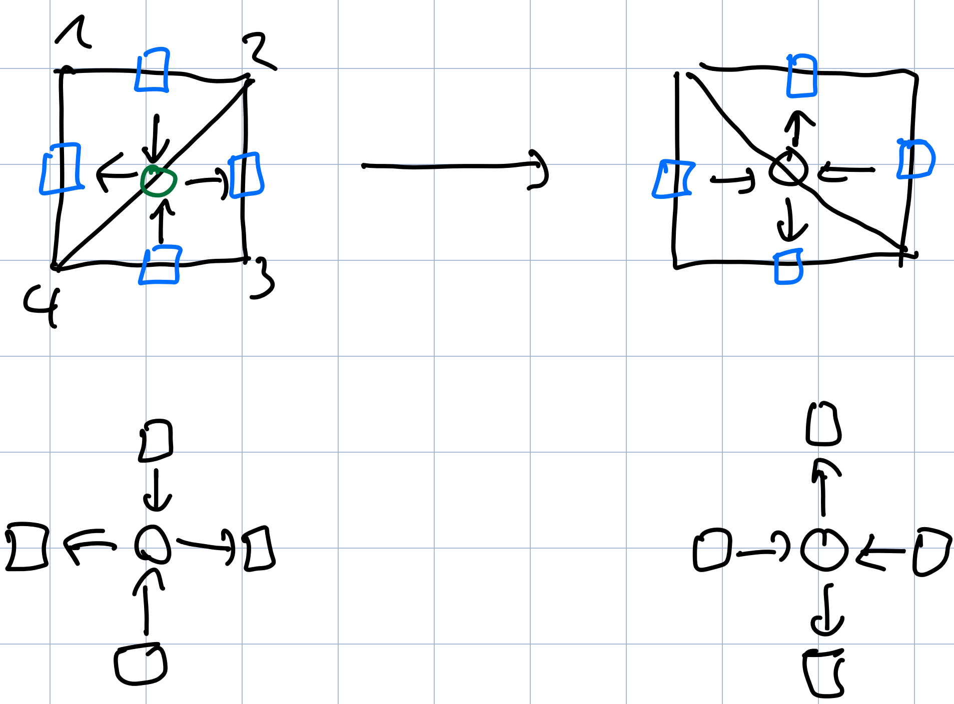
- Label the sides with frozen vertex

The diagonals will non frozen vertex

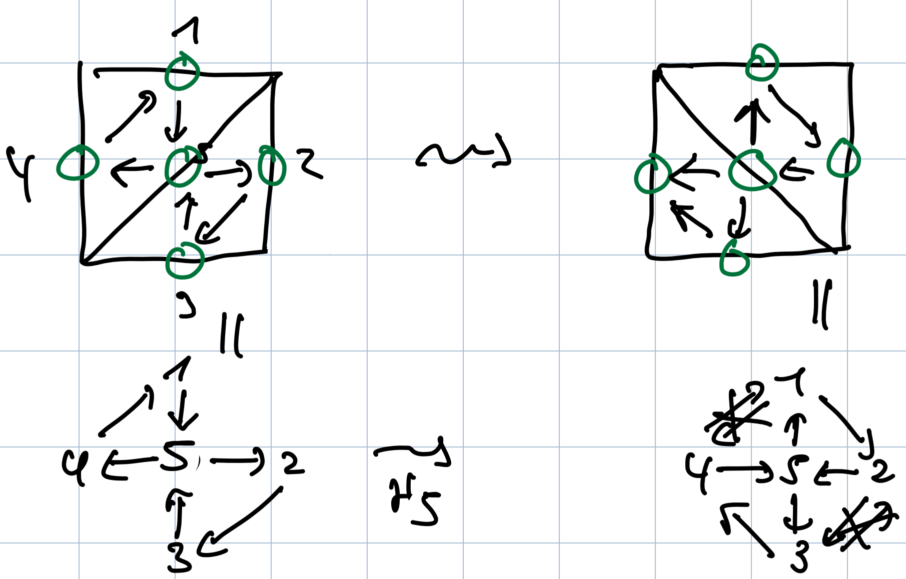
- If two diag, a side belong to a triangle add an edge between them following clockwise order

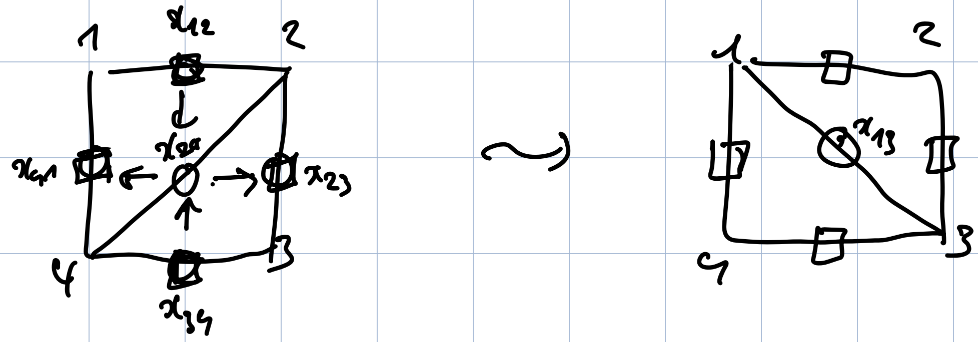
We denote by $Q(T)$ the corresponding quiver

- Ex



Generic

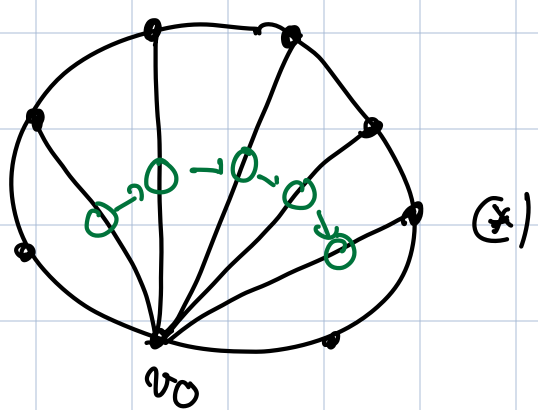


- 

$$\boxed{x_{13}\,x_{24} = x_{12}\,x_{34} + x_{23}\,x_{41}}$$

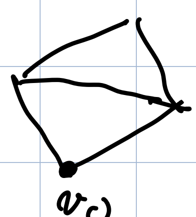
(Ptolemy's thm for cyclic quadrilaterals).

- Ex

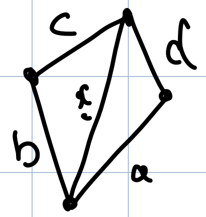
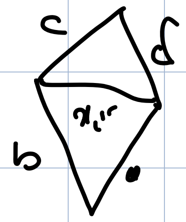


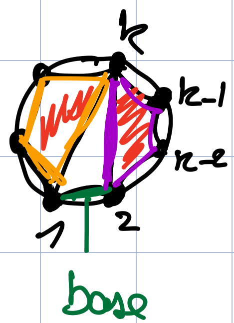
- Any two triangulations are related by a sequence of flips

Indeed we show that any triangulation can be modified to (*)
if less than $n-3$ vertices connected to v_0, then can find a square



flip in it strictly increase the number of edges connected to v_0.

-   $x_i x_i' = ac + bd$

- 

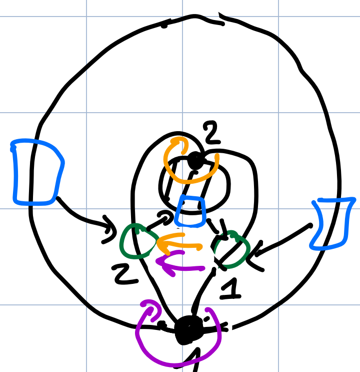
So a triangulation is a root, left and a right triangulation or the empty triangulation

So generating function is $G = 1 + xG^2 \iff xG^2 - G + 1 = 0$

$\Delta = 1 - 4x$ solutions $\frac{1 \pm \sqrt{1-4x}}{2x}$ only one with positive values

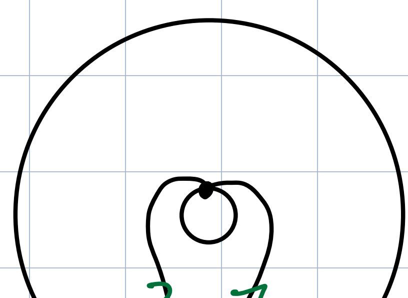
$$\Rightarrow G = \frac{1 - \sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n.$$

Ex annulus surface triangulation

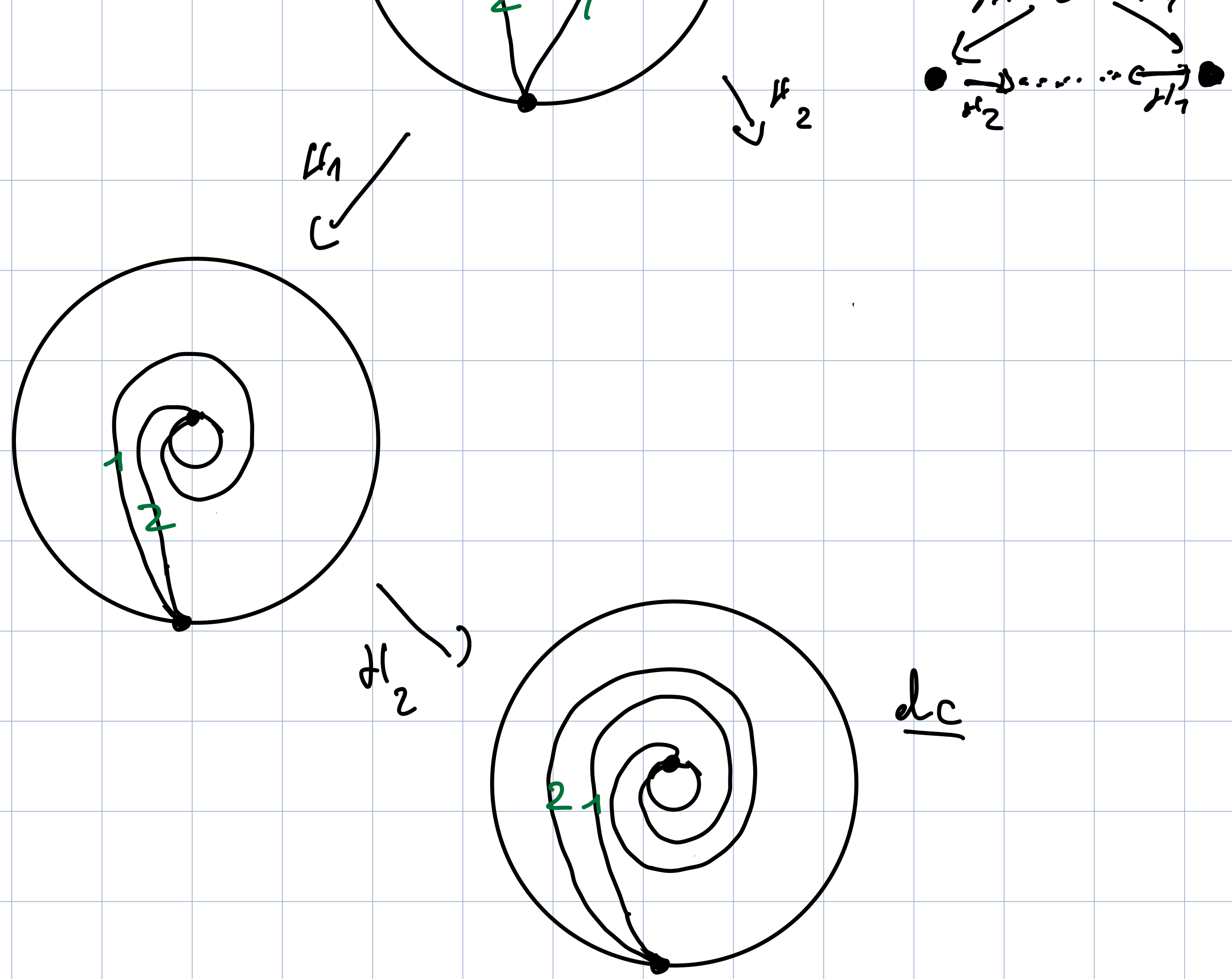


hence the corresponding quiver is $1 \rightrightarrows 2$ the Kronecker quiver.

<u>Mutations</u>



Exchange graph



H1
H2
1
2
H2
H1
H2
1
2
H2
2 1
dc